Interaction between propagation and consensus on module networks

Fang Liu[1*], Yi Zheng[2]

[1]*Jiangsu Open University, Nanjing, 210036, China*

[2]*Nanjing University of Posts and Telecommunications, Nanjing, 210003, China*

Received 1 March 2014, www.cmnt.lv

Abstract

The effect on spreading dynamics induced by collective consensus is investigated in this paper, on the basis of epidemic model for spreading dynamics and phase synchronization model for collective consensus. The interaction between propagation and collective synchronization is explored by theoretical analysis and numerical simulation. We found that the spreading dynamics and synchronizations are highly affected with the increasing of coupling strength on sparse links between communities. With the level of collective consensus about awareness increasing, both oscillation amplitude and mean prevalence are suppressed, and the inter-community coupling strength have great effect on time spent on reaching consensus.

Keywords: phase synchronization, module network, information spreading, collective behaviour

1 Introduction

In the real-world networks, there exists collective synchronization phenomenon almost everywhere, which has aroused wide concern recently and has been researched widely, and a population of coupled oscillators and a network model describing the link and interaction between individuals are applied to characterize this phenomenology. These two ingredients trigger various dynamic behaviours. In particular, the collective synchronization modelled by epidemic process varies with the oscillator network. The earliest discussion of the periodic oscillations of epidemic model focuses on small-world networks [1]. They found that with the rewiring probability slowly increasing, the number of infected individuals will eventually reach a state of apparent periodic oscillations with the time series instead of fluctuating at a fixed point originally. By using a modified SIR model, the epidemic threshold pc for case of mean degree k=2 on small world networks has been investigated [2]. According to the above works, we can start our research about measuring the small-world effect from the perspective of the dynamics of system, beyond the limit of the topological analysing method.

Recently, model of networks that consist of inter-connected modules have been extensively investigated in collective synchronization research field. Yan and Fu propose a scale-free module network with adjustable community strength, to explore collective synchronization induced by the Susceptive-Infected-Recovery-Susceptive epidemic model [3]. They found that small module strength induces better global synchronization and there exists a critical point where phase transition occurs. Using the same module network model, effects of community structure on the phase synchronization has been investigated for Kuramoto model [4]. They found that there exist a solution region, in which system have worse synchronization than isolated community graphs. In reference [5], the author used the Susceptive-Infected-Recovery-Susceptive (SIRS) epidemic model to figure out the influence of community structure on the temporal dynamics of the epidemic spreading, and the order parameter was introduced to demonstrate the synchronization and related to the phase of nodes at the present time. After the study, they found that small community structure induces better synchronization and there exists a critical point where a phase transition occurs.

The above theoretic research convincingly demonstrates the relationship between community structure and collective synchronization. However, other factor like the human collective behavior, such as leaving the prevalence area, awareness mechanism, also plays an important role in the spreading dynamics, and the correlational research is lacking. In the real world, when an infectious disease breaks out, people will take some safeguard measures to reduce the risk of being infected sooner or later. When the majority of people's awareness is formed, the epidemic spreading will be greatly blocked and the collective synchronization induced by it may be influenced similarly. How the awareness mechanism works in the synchronization phenomena is what we want to explore. The authors in reference [6] researched the interplay between awareness and epidemic dynamics by a continuous mean-field (MF) model. They started their work from three perspectives: local awareness, global awareness and contact awareness. The individual awareness influences the epidemic spreading by influencing the admission rate,

* ***Corresponding author's*** e-mail: liufang@jstvu.edu.cn

and the larger the admission rate is, the weaker the individual is, and then it is easier for it to be infected. For node *i*, the author used a specific function to denote its admission rate, which involves the total number of infected individuals. However, we find that it is always hard to acquire the total number of infected individuals in the part of global awareness immediately and precisely though the number of infected neighbours can be acquired spontaneously. Therefore, we searched for much literature and found that the research on consensus and phase synchronization in reference [7, 8] is suitable to be applied to solve the above problem.

In reference [8], based on the famous Kuramoto model and the general framework proposed in the author's previous research, a set of three differential equations was proposed to build a dynamics system, which denotes the changes of an individual's phase, the epidemic prevalence and the coupling strength of the network structure. The synchronization was measured by the phase synchronization error, i.e., the consensus. At the outbreak of some disease, people can meet a consensus about the severity of the disease by communicating online or other means, so the global data needn't be acquired which may not be precise even if it is acquired immediately. After research, they found that the effect of network structure on the spreading process while to enhance the awareness to collective behaviour is an effective method to control the epidemic spreading.

In reference [9-11], a novel control method to improve the system synchronization are applied to research field of power grid, consensus networks. The outperforming of system can be achieved by sparse wide-area control, just like the centralized control, while with lower communication cost. Inspired by the idea of sparse inter-area control, we try to answer the following questions with our work: how the sparse links between communities affect the collective synchronization with scale and speed, and the subsequent interaction between spreading dynamics and collective phase synchronization is or not affected by the module structure, which mean the structure with dense links in community and sparse links between communities.

In this paper we investigate the interplay of phase synchronization and epidemic-modeled dynamics in modular networks. The considered modular networks consist of densely connected modules, each with the Scale-Free network structure.

2 Model

Many real networks have some common properties, for example, the small-world, scale-free and clustering properties. Scale-free means that there are some nodes with very large degrees, act as hubs in the whole graph. And this feature was not gathered by the basic ER random graphs. To generate scale-free graphs, there are several models in theoretical study. Yan proposed a growth model as follows [3]: starts from n community cores; each core contains m0 fully connected nodes. Initially, there are no connections among different community cores. At each time step, to each community core, one node is added. Thus, there are in total n new nodes being added in one time step. Each node will attach m edges to existing nodes within the same community core, and simultaneously m-n edges to existing nodes outside this community core. As results, the algorithm creates a network with adjustable ratio of edges intra-community to the total edges, together with a degree distribution following a power-law by constantly adding new nodes. The definition of the community strength, as follows:

$$Q = \sum_{1}^{c} \left[\frac{l_s}{L} - \left(\frac{d_s}{2L} \right)^2 \right], \tag{1}$$

where c is the number of the modules, L is the overall total number of edges, l_s is the total number of edges in the module, d_s is cumulative sum for the node degree within the module.

The main purpose of most research done in complex network is to figure out systems built on networks and the interaction between the structure of the network and the evolution of the system. These systems can be shown using dynamical processes taking place on networks. And in this paper, we use Kuramoto model to describe the behavior of the oscillators. Kuramoto model, a fully nonlinear model used to describe synchronization, has the following governing Equation:

$$\dot{\theta}_i(\mathrm{t}) = \omega_i - \rho \sum_j a_{ij} \sin(\theta_i - \theta_j), \tag{2}$$

where the parameter ρ determines the coupling strength, a_{ij} is the adjacency matrix, θ_i is the phase of oscillator, ω_i denotes intrinsic natural frequency of oscillator i, N is the number of whole population. The simulation are performed under the assumption that the random variable θ_i and ω_i are uniformly distributed in $[0,2\pi]$ and $[-0.5, 0.5]$, respectively. The nature frequencies in each node induce oscillators, while the coupling strength always synchronize the phase of the system. When the coupling strength overcome the centrifugal force induced by nature frequency, the synchronized order parameter increases. To quantify the synchronized states we use the relevant order parameter:

$$\delta = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \right|. \tag{3}$$

Focusing on the purpose to explore the effect of sparse control on inter-community links, we build our phase synchrony model based on Kuramoto model. The phase dynamics of the system with time-varying intra-community coupling strength $\beta(t)$ and time-varying inter-community coupling strength $\alpha(t)$.

$$\dot{\theta}_i(m,t) = \omega_i - \beta_m(t)\sum_{j=(m-1)N_c+1}^{mN_c} a_{ij}\sin(\theta_i(t)-\theta_j(t)) - p\sum_{n\in c,n\neq m}\alpha_{mn}(t)\sum_{k=(n-1)N_c+1}^{nN_c} a_{ik}\sin(\theta_i(t)-\theta_k(t)), \quad (4)$$

where m denotes the community number of node i, β is the coupling strength intra-community, N_c is the population number of each community, $N=N_c\times C$, α_{nm} is the inter-community coupling strength between community m and n. $p\in(0,1)$ is a magnifying parameter.

$$\beta_m(t) = 1-\beta_0(1-I_m(t)), \quad (5)$$

where $\beta_0\in[0,1]$ is a constant parameter, $I_m(t)$ is the proportion of infected individuals in the population of community m. The intra-community coupling strength in community m is $\beta_m(t)\in[1-\beta_0,1]$, and varying with $I_m(t)$. The assumption shown by Equation (5) is as follows: in a community with dense links, individuals are easy to obtain the data of epidemic prevalence in its own community and change their intra-community coupling strength according to the severity of the epidemic outbreak. For example, when an infectious disease breaks out, individuals in same groups will reach consensus more urgently, when the epidemic prevalence level is high. At the same time, the intra-community coupling strength should always exist even the epidemic prevalence level is very low.

$$\dot{\alpha}_{mn}(t) = 1-\frac{1}{2N_c}\left|\sum_{j\in m\cup n} e^{i\theta_j}\right|. \quad (6)$$

The assumption shown by Equation (6) is as follows: the change rate of inter-community coupling strength on the sparse links between two communities is negative relate to the level of collective synchronization in these two communities. That means, when individuals found the phase error between their own community and neighbor community is increasing, then the coupling strength will increase to suppress the gap between two communities. This phenomenon widely exists in the real world.

The spreading dynamics are modeled in our work by epidemic SIRS model, and the effect of consensus is denoted as the vulnerability of the nodes in graph. That means, with a high level of phase synchronization, which means high level of consensus, then high level of protect measure will be adopted. This assumption is based on the epidemic model, while effect of consensus is not necessarily negative on information spreading.

$$\dot{S}_k(t) = -\lambda\psi(t)S(t)I(t)+\mu R(t), \quad (7)$$

where λ is the contact rate for spreading, ψ is the vulnerability of the node with degree k, μ is the immune-failure rate for epidemic SIRS models, $T_r=1/\mu$.

$$\dot{I}(t) = \lambda\psi(t)S(t)I(t)-rI(t), \quad (8)$$

where γ is the recovery rate for SIRS models, $T_i=1/\gamma$.

$$\psi(t) = 1-(1-\varepsilon)\delta, \quad (9)$$

where $\varepsilon\in[0,1]$ is a constant parameter, the vulnerability parameter is $\psi(t)\in[\varepsilon,1]$.

On the basis of the set of differential Equations (7-9), we propose our model to explore the role that awareness plays in the synchronization mainly by adding the effect of consensus as awareness to our epidemic model. For example, when a panic spreads, many individuals will decrease their contacts for the precaution, while their communication about the epidemic or rumor is keeping unblocked through Internet, telephone.

3 Results and discussion

Here we get the prevalence of epidemic and relevant order parameter by simulation over 100 time steps, to make sure that curve of the system is in a steady state. Meanwhile we assume the population size is N=2000, number of communities C=5, that means there 400 individuals in one community, and various Q to control the number of links intra- and inter- community.

Figures 1-2 displays the effect of order parameter δ on infected proportion $I(t)$ with fixed intra-community coupling strength β and various inter-community coupling strength p. Overall, it is easy to find that with high degree of inter-community coupling strength, oscillation amplitude and mean prevalence are suppressed. In Figure 2 the system converge to a weak oscillating. With high degree of inter-community strength p, global synchronization is reaching its peak region in a slower speed, which is a quite counterintuitive result.

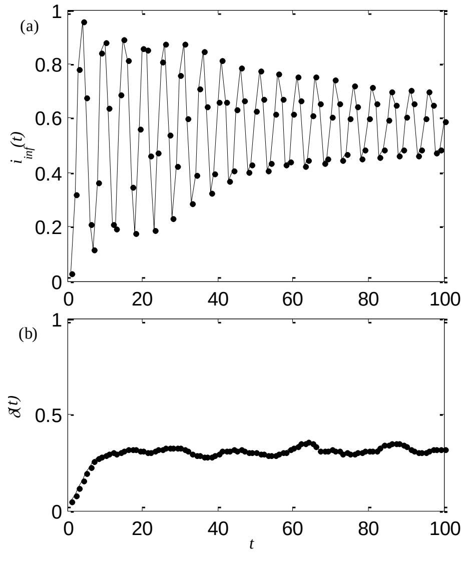

FIGURE 1 The change of (a) epidemic prevalence $I(t)$ and (b) phase synchronized order parameter $\delta(t)$ with β_0=0.2, λ=0.2, p=0.5, Q=0.80 T_i=2, T_r=2

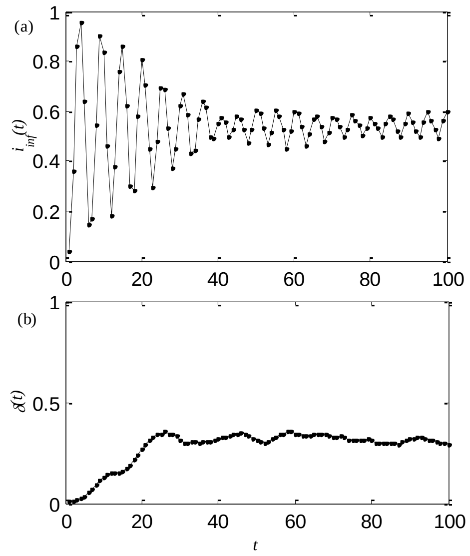

FIGURE 2 The change of (a) epidemic prevalence $I(t)$ and (b) phase synchronized order parameter $\delta(t)$ with β_0=0.2, λ=0.2, p=0.9, Q=0.80, T_i=2, T_r=2

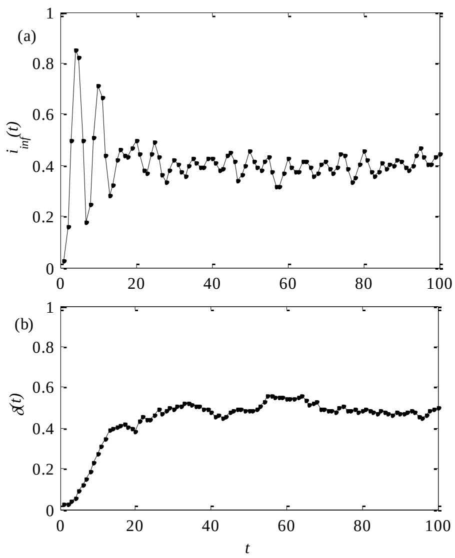

FIGURE 3 The change of (a) epidemic prevalence $I(t)$ and (b) phase synchronized order parameter $\delta(t)$ with β_0=0.2, λ=0.2 p=0.9, Q=065, T_i=2, T_r=2

Results in Figure 3 show the epidemic prevalence and consensus on module network with same parameters with Figure 2 but community strength Q=0.65. The change law in Figure 3 clearly shown that, with the decreasing of Q, which means more links changing from intra-community edges to inter-community edges, the control effect induced by phase synchronization are taking more quickly. There only two peaks before the curve of $I(t)$ hit the stable line. Combined with the results from Figure 2, a reasonable explain is as follows: lower Q means the connectivity of inter-community increased, high p means the coupling strength between a pair of communities increased. Note that in the situation of high p and high Q, where the strong coupling strength with poor connectivity between communities, and due to this, results in lower speed to achieve synchronization.

To validate our results from above, we simulated the same change of Q and p in SIS models. With assuming the T_r=0, the periodic oscillations are fade out. The difference between the change law in Figures 4-6 shown the same rule as Figures 1-3. More links between modules, more quick the response speed. The epidemic prevalence is greatly affected by phase synchronization, the length of T_r and T_i, but community strength Q. The Strengthen of inter-community coupling strength have great effect on the stable state of epidemic, and delayed the reaching of consensus.

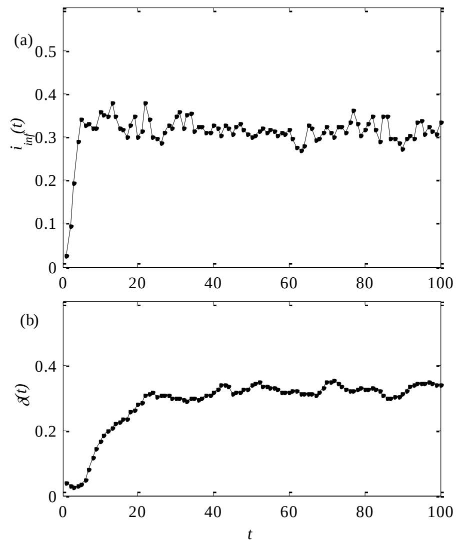

FIGURE 4 The change of (a) epidemic prevalence $I(t)$ and (b) phase synchronized order parameter $\delta(t)$ with β_0=0.2, λ=0.2, p=0.5, Q=0.80, T_i=2, T_r =0

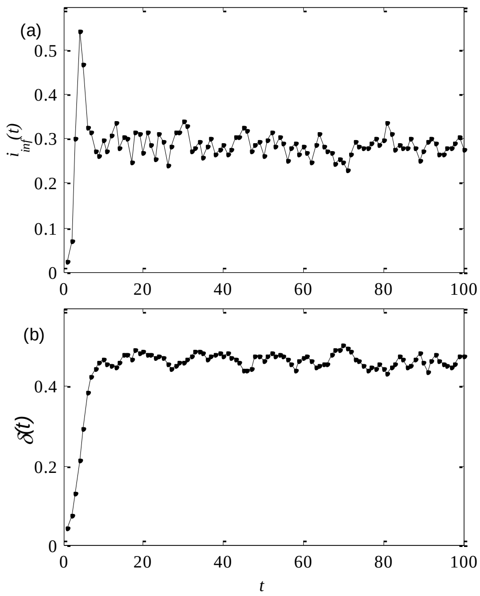

FIGURE 5 The change of (a) epidemic prevalence $I(t)$ and (b) phase synchronized order parameter $\delta(t)$ with β_0=0.2, λ=0.2, p=0.5, Q=0.65, Ti=2, Tr=0

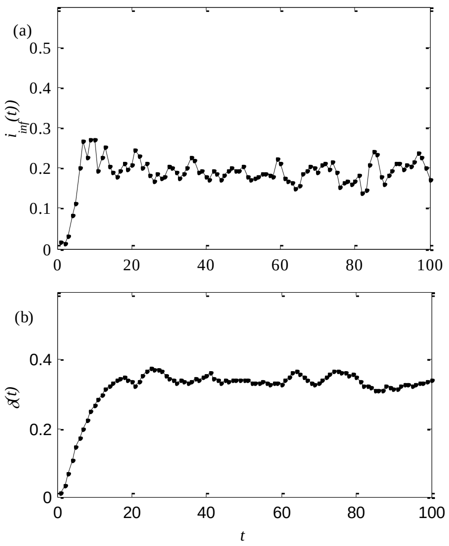

FIGURE 6 The change of (a) epidemic prevalence *I(t)* and (b) phase synchronized order parameter *δ(t)* with β_0=0.2, λ=0.2, p=0.9, Q=0.65, T_i=2, T_r=0

4 Conclusions

We have investigated the influence of the coupling strength of inter- and intra-community on global synchronization induced by the Kuramoto model. These models can be argued to capture the social collective behavior better than the original module models, since time-varying inter-community coupling considered. Our models can be extended to discuss the effect of cluster coefficient on dynamics, because of the design of model are applicable to analyze the coefficient. High community strength Q with sparse links between modules, suppressed the spreading and phase synchronization procedure. Note that this results a fade out of phase synchronization even in high coupling strength. When a high inter-community coupling strength combined with low community strength Q, there is a clear decrease in spreading prevalence and an obvious speed advantage to hit the stable state. However, that also corresponds the most cost expensive scheme, because of the increase of edges and coupling between modules. From the results from our work, we can conclude that the scheme place extra emphasis on inter-community coupling strength has the cost advantage and the scheme place extra emphasis on inter-community connectivity has the speed advantage.

Acknowledgments

We acknowledge the support from the National Natural Science Foundation of China (grant No.50875132, 60573172), the Natural Science Foundation of Jiangsu, China (grants No.BK2012082) and the City Vocational College of Jiangsu 12th Five-year Plan Key Subject (No.12SEW-Z-005).

References

[1] Kuperman M, Abramson G 2001 *Physical Review Letters* **86**(13) 2909-12
[2] Zanette D H 2001 *Physical Review E* **64**(5) 050901
[3] Yan G, Fu Z-Q, Ren J, Wang W-X 2007 *Physical Review E* **75**(1) 016108
[4] Zhou T, Zhao M, Chen G, Yan, G, Wang B-H 2007 *Physics Letters A* **368**(6) 431-4
[5] Girvan M, Newman M E 2002 *Proceedings of the National Academy of Sciences* **99**(12) 7821-26
[6] Li K-Z, Fu X-C, Small M,Ma Z-J 2011 *Chaos* **21**(3) 03311
[7] Wu Q C, Fu X C, Small M, Xu X J 2012 *Chaos* **22**(1) 013101
[8] Li K-Z, Ma Z-J, Jia Z, Small M, Fu X-C 2012 *Chaos* **22**(4) 043113
[9] Dorfler F, Jovanovic M, Chertkov M, Bullo F 2014 *IEEE Transactions on Power Systems* **29**(5) 2281-91
[10] MirTabatabaei A, Bullo F 2012 *SIAM Journal on Control and Optimization* **50**(5) 2763-85
[11] Pasqualetti F, Borra D, Bullo F 2014 *Automatica* **50**(2) 349-58

Authors

Liu Fang, 14.11.1976, Anhui, China.

Current position, grades: lecturer of Jiangsu Open University, Nanjing, China.
University studies: MSc. degrees from Nanjing University of Aeronautics and Astronautics.
Scientific interest: epidemic dynamics on complex networks, cascade in complex networks and power grids.

Zheng Yi, 15.03.1977, Shandong, China.

Current position, grades: Phd candidate of Nanjing University of Posts and Telecommunications, China.
University studies: M.S. degrees from Nanjing University of Aeronautics and Astronautics.
Scientific interest: epidemic dynamics on complex networks, cascade in complex networks and power grids.